ARTICLE II.

Supplementary Note on the Construction and different Forms of the Magic Cyclovolute.
By E. Nulty. Read December 6, 1844.

THE arrangement of numbers, denominated a *Magic Cyclovolute*, and published in the fifth volume of the Society's Transactions, 1835, has for its basis a *perfect magic square*, formed of the sixty-four integral numbers, 1, 2, 3 . . . 64. This peculiar square essentially differs, in its construction, from the basis of the magic circle, which must be viewed as *imperfect*, so far as respects an inequality which subsists between the sums of the numbers along its horizontal and vertical rows, and those constituting its principal diagonals. The perfect form which we here propose to construct may be regarded as compound, and as generated from two others: one consisting of binary combinations taken from the series of eight numbers, 1, 2, 3 . . . 8; the other, of multiples by *eight* of the different terms of the corresponding series, 0, 1, 2 . . . 7, and which, being joined by addition to the preceding, will evidently give the several terms of the first series, 1, 2, 3 . . . 64. In order to exhibit these component squares in reference to the particular arrangement published, let us take the two binary combinations (1, 8) and (2, 7), which being placed in vertical and horizontal rows, will form the two following elementary magic squares:

1	2	8	7
8	7	1	2
1	2	8	7
8	7	1	2

. . . . (*a*)

1	8	1	8
7	2	7	2
8	1	8	1
2	7	2	7

. . . . (*a'*)

In like manner from the two binary combinations (3, 6) and (4, 5), let us form the analogous perfect squares,

3	4	6	5
6	5	3	4
3	4	6	5
6	5	3	4

. . . . (*b*)

3	6	3	6
5	4	5	4
6	3	6	3
4	5	4	5

. . . . (*b'*)

in which, and the preceding, the upper row of the first is made the descending diagonal of the second; and the upper row of the second is the leading vertical row of the first.

If we repeat these partial squares, the first and third, (a) and (b), vertically; and the second and fourth, (a') and (b'), horizontally, we shall obtain by their junction the following complete magic squares, (A) and (B), which as respects their diagonal and leading rows, are constituted in a manner similar to their components, (a), (a') and (b), (b'):

(A)

1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4
1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4
1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4
1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4

(B)

1	8	1	8	1	8	1	8
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1
2	7	2	7	2	7	2	7
3	6	3	6	3	6	3	6
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
4	5	4	5	4	5	4	5

Of these fundamental squares, which we prefer in their present forms, either may be considered as a *primitive*, the other its *derivative*. To suit our present purpose we shall take (A) as primitive, and we shall mentally diminish its several numbers by unity in accordance with the series 0, 1, 2...7. By multiplying each of the remainders thus obtained by eight, and adding their products in successive order to the corresponding numbers, similarly posited in the derivative (B), we shall readily construct the perfect magic square (AB), of which all the rows, whether vertical, horizontal, or diagonal, amount to 260; and which is obviously composed of four perfect magic squares, having their diagonals and other rows each equal to 130, the common amount of every four adjacent numbers in the entire square.

(AB)

1	16	57	56	17	32	41	40
63	50	7	10	47	34	23	26
8	9	64	49	24	25	48	33
58	55	2	15	42	39	18	31
3	14	59	54	19	30	43	38
61	52	5	12	45	36	21	28
6	11	62	51	22	27	46	35
60	53	4	13	44	37	20	29

The arrangement here given is remarkable. It will remain perfectly magic when any number of its vertical rows are removed in order, and placed in succession after the last; and also when any number of its upper or lower horizontal rows undergo a similar displacement. Such permanency in the magic structure of (AB) follows from the peculiar mode in which the binary combinations taken from the series 1, 2, 3...8, are disposed in the fundamental forms (A) and (B); and hence we are immediately led to the constant result of the several principal and secondary rings of the magic cyclovolute.

As to the general result common to all the volutes and the particular property, inserted in the Bulletin of the society, (No. 13, 1840), in reference to the sixteen semi-volutes, reckoned from the extremities of the principal diameters AA', BB', and the corresponding points a, a', and b, b', they are a consequence of (AB) rendered perfectly magic, as well as the four squares which compose it. With these semi-volutes, we may also notice the sixteen semi-radii in the drawing. The four numbers in each of them with half the auxiliary 12 amount to 180; and a like result will always hold good for every four numbers taken in adjacent pairs from the centre, or, from the remote extremities of any radius. These supplemental properties render peculiar the *form* of the drawing as now considered, and give it every possible generality.

The square (AB) will directly bring to view the arrangement adopted in the drawing. We have merely to increase each of its numbers by 11, in order to adapt it to the amount 360, employed by the author of the magic circle, instead of 260 above-mentioned; and to place the same horizontal rows of the corresponding magic square in succession round the principal rings of the cyclovolute, above and below the principal diameter AA'.

Let us now determine the number of magic cyclovolutes that may be constructed by arranging differently the terms of the series 1, 2, 3 ... 64. With this view, let us return to the first squares (a), (a'), and let them be combined with similar squares formed from the reverse binary combinations (5, 4) and (6, 3) and constituting (c) and (c') here given:

5 6	4 3
4 3	5 6
5 6	4 3
4 3	5 6

. . . (c)

5 4	5 4
3 6	3 6
4 5	4 5
6 3	6 3

. . . . (c')

We shall then obtain, as before, the two new fundamental squares (A') and (B'):

1 2	8 7	5 6	4 3
8 7	1 2	4 3	5 6
1 2	8 7	5 6	4 3
8 7	1 2	4 3	5 6
1 2	8 7	5 6	4 3
8 7	1 2	4 3	5 6
1 2	8 7	5 6	4 3
8 7	1 2	4 3	5 6

. . . . (A')

1 8	1 8	1 8	1 8
7 2	7 2	7 2	7 2
8 1	8 1	8 1	8 1
2 7	2 7	2 7	2 7
5 4	5 4	5 4	5 4
3 6	3 6	3 6	3 6
4 5	4 5	4 5	4 5
6 3	6 3	6 3	6 3

. . . . (B')

which are precisely similar to (A) and (B); and may be employed for like purposes. If we use (A') as a primitive in the manner of (A) already treated, we shall find from it and the derivative (B') the compound square $(A'\,B')$, in which and any similar arrangement, attention must be paid to the order of the letters A', B', as indicative of the particular mode of combination. Besides the two compound squares (AB), $(A'\,B')$, thus obtained, we may in a similar manner obtain the squares (BA), $(B'\,A')$, and also the four additional squares (AB') $(A'B)$, and (BA') $(B'A)$; so that the elementary forms (a) and (a') considered as invariable, will thus lead to *eight* compound magic squares, having all the same properties. It is also evident that the binary combinations (1, 8), (3, 6), and (1, 8), (5, 4), will also furnish squares analogous to (a), (a'), by which and the remaining combinations (2, 7), (4, 5), and (2, 7), (6, 3), we shall be enabled to form *sixteen* new compound magic squares, which, with the preceding eight, amount to twenty-four different arrangements, as given at the end of this note.

We may immediately derive other forms from these, simply by interchanging the first and second vertical rows, the third and fourth, &c. By virtue of the forms $(A,)$ (A'), the resulting squares will be similar to (AB); and, in this way, the number of arrangements becomes forty-eight. These are the only changes that can be made in the vertical rows, so as to give essentially different magic cyclovolutes. To obtain other arrangements we must change the order of the horizontal rows, which, commencing with the upper, we shall briefly denote by the numbers 1, 2, 3, 4 || 5, 6, 7, 8. If we interchange every two of these, the resulting form will be 2, 1, 4, 3 || 6, 5, 8, 7; and taking also the second direct form, 1, 8, 3, 6 || 5, 4, 7, 2, which is equally applicable, there will, in like manner,

result 8, 1, 6, 3 || 4, 5, 2, 7. These may have each four of their rows inverted, relatively to their partial squares, in the order 4, 3, 2, 1 || 8, 7, 6, 5, &c. Every magic square (C) of the forty-eight above considered will thus give seven others, (C_1), (C_2), . . . (C_7); and accordingly, the total number of different arrangements similar to (C) will become $48 \cdot 8 = 384$. We may obviously invert each of these arrangements in the circular distribution of their rows, round the principal rings of the drawing, and we shall thus have it in our power to construct 768 corresponding magic cyclovolutes.

In the formation of all the arrangements hitherto determined, we have kept in view the various properties ascribed to the magic cyclovolute, based on the square (AB). By merely excluding the consideration of that particular property, connected with the numbers taken in pairs along the radii of the drawing, as already mentioned, an additional group of magic cyclovolutes may be constructed, which with the 768 preceding, will amount to 6044; and other groups would result in case the semi-volutes, or the semi-radii, or both, were left unnoticed as minor properties. It would be somewhat tedious, and of no particular interest, to ascertain the precise number of forms corresponding to each of these limited hypotheses. We shall therefore omit any results of this kind, and proceed directly to investigate the total number of unclassified magic cyclovolutes, including the 6044 above determined, and all the general properties originally enumerated.

Let us then return to the form 1, 2, 3, 4 || 5, 6, 7, 8, adopted for any magic square (C), and let us observe that no two *odd*, or two *even* numbers, indicative of the horizontal rows, can be disposed and arranged successively in any derivative form, considered as the basis of a magic cyclovolute. The reason of this will appear from the square (AB), to which all additional forms must preserve a like structure. Attending to this essential condition, the number of combinations of the *four* odd rows, 1, 3, 5, 7, or of the *four* even rows, 2, 4, 6, 8, taken in pairs, is $\frac{1}{2}(4 \cdot 3) = 6$; and these combined in *fours*, of which two are *odd* and two *even*, will give $6 \cdot 6 = 36$ forms, and consequently eighteen different arrangements analogous to 1, 2, 3, 4 || 5, 6, 7, 8. But any of these forms so constructed, as for instance, the preceding, admits of *four* changes with respect to the rows 1, 2, 3, 4, and also *four* for the rows 5, 6, 7, 8. These rows may therefore be combined $4 \cdot 4 = 16$ ways; and taking the inverse form 4, 3, 2, 1 || 8, 7, 6, 5, as equally admissible, we shall have thirty-two arrangements, instead of the single primitive form 1, 2, 3, 4 || 5, 6, 7, 8; and the eighteen different arrangements previously determined will generate the number $18 \cdot 32 = 576$. By viewing these combinations in a less specific manner, the number just found will result as the square of ($4 \cdot 3 \cdot 2 \cdot 1$), the permutations of either the odd or even rows. We shall thus increase the forty-eight magic squares first considered to $48 \cdot 576 = 3 \cdot 96^2$ or 27648, which may be all obtained from the series of sixty-four numbers, 1, 2, 3 . . . 64, and converted into magic cyclovolutes. The total number of these remarkable arrangements, with all the leading properties in my paper in the Transactions of the Society, therefore, amount to $6 \cdot 96^2 = 55296$.

In connexion with our subject, we shall here bring to notice a *new* imperfect magic square, analogous to that adopted by Dr. Franklin in the construction of his magic circle; but which so far generalizes it, as to include the particular property of the numbers taken in pairs along the several radii, as already mentioned in case of the cyclovolute. It is

singular that this extension should have hitherto escaped notice, and that the magic circle in its present form should yet admit of improvement. To establish this point let us form the two elementary squares (d) and (d') thus:

1	2	7	8
8	7	2	1
2	1	8	7
7	8	1	2

. . . . (d)

1	8	1	8
7	2	7	2
2	7	2	7
8	1	8	1

. . . (d')

and also similar squares, (e), (e') and (f), (f') by means of the binary combinations (3, 6), (4, 5) and (5, 4), (6, 3). The repetition of these will give *four* fundamental squares, (D), (E) and (D'), (E'), the second and fourth of which are imperfectly magic; and from these by combination, we shall obtain the *eight* imperfect magic squares (DE), ($D' E'$), &c., the first of which is here subjoined. The manner in which these *eight* are increased to *forty-eight*, we have already explained, and taking the *three* combinations 1, 2, 3, 4 ‖ 5, 6, 7, 8, and 1, 4, 7, 6 ‖ 5, 8 3, 2, and 1, 2, 7, 8 ‖ 5, 6, 3, 4, we shall immediately find by their changes the number $48 . 32 = 1536$ magic circles with our additional properties; $48 . 160 = 7680$ of Dr. Franklin's limited form; and the total number without distinction of properties will result as before $6 . 96^2 = 55296$.

(DE).

1	16	49	64	17	32	33	48
63	50	15	2	47	34	31	18
10	7	58	55	26	23	42	39
56	57	8	9	40	41	24	25
3	14	51	62	19	30	35	46
61	52	13	4	45	36	29	20
12	5	60	53	28	21	44	37
54	59	6	11	38	43	22	27

The principles by which we have been guided to the various results in this paper, we judge of considerable moment in regard to the magical combination of numbers. They may be readily applied to the extensive series of 256 numbers 1, 2, 3 . . . 256, first considered in a magic form by Dr. Franklin, and left combined by him in an imperfect magic square, similar in properties to the base of his magic circle. This he no doubt intended as a generalized base of an enlarged magic circle, but owing to an oversight in two or three of the numbers, it would not be applicable. In Dr. Hutton's Mathematical Dictionary, which I have lately consulted, this imperfection is said to have been noticed by Mr. Dalby, "first Professor at the Royal Military College;" and to this ingenious mathematician is ascribed the formation of a remarkable magic square of *sixteen* perfect magic squares, including the above series; and which is given in the work just cited. Without the slightest intimation of Mr. Dalby's square, or that of any other person, I had been led to several analogous arrangements. My method is that employed in case of the perfect magic square (AB). I formed from the binary combinations (1, 16), (2, 15), (3, 14), (4, 13) and (5, 12), (6, 11), (7, 10), (8, 9), the elementary squares

1	2	16	15	3	4	14	13
16	15	1	2	14	13	3	4
1	2	16	15	3	4	14	13
16	15	1	2	14	13	3	4
1	2	16	15	3	4	14	13
16	15	1	2	14	13	3	4
1	2	16	15	3	4	14	13
16	15	1	2	14	13	3	4

. . . . (g)

5	6	12	11	7	8	10	9
12	11	5	6	10	9	7	8
5	6	12	11	7	8	10	9
12	11	5	6	10	9	7	8
5	6	12	11	7	8	10	9
12	11	5	6	10	9	7	8
5	6	12	11	7	8	10	9
12	11	5	6	10	9	7	8

. . . . (g')

the descending diagonal rows of which were made the leading vertical rows of two similar squares, (h), (h'); and from the reverse combinations (9, 8), (10, 7), (11, 6), (12, 5), I constructed two additional squares (i), (i'). It is obvious that I had it in my power to form other analogous squares, by taking in a different order the combinations here given. From the squares thus made, I obtained the fundamental squares (G), (H), (G'), (H'); and from these, as before, the compound perfect arrangements (GH), (HG), &c. To serve as a comparison with Mr. Dalby's, and to notice its remarkable properties in my own way, I shall present the second combination, (HG).

New perfect magic square, including sixteen perfect magic squares, formed of the series of 256 *numbers*, 1, 2, 3 . . . . 256.

1	242	16	255	3	244	14	253	5	246	12	251	7	228	10	229
240	31	225	18	238	29	227	20	236	27	229	22	234	25	231	24
241	2	256	15	243	4	254	13	245	6	252	11	247	8	250	9
32	239	17	226	30	237	19	228	28	235	21	230	26	233	23	232
33	210	48	223	35	212	46	221	37	214	44	219	39	216	42	217
208	63	193	50	206	61	195	52	204	59	197	54	202	57	199	56
209	34	224	47	211	36	222	45	213	38	220	43	215	40	218	41
64	207	49	194	62	205	51	196	60	203	53	198	58	211	55	200
65	178	80	191	67	180	78	189	69	182	76	187	71	184	74	185
176	95	161	82	174	93	163	84	172	91	165	86	170	89	167	88
177	66	192	79	179	68	90	77	181	70	188	75	183	72	186	73
96	175	81	162	94	173	83	164	92	171	85	166	90	169	87	168
97	146	112	159	99	148	110	157	101	150	108	155	103	152	106	153
144	137	129	114	142	125	131	116	140	123	133	118	138	121	135	120
145	98	160	111	147	100	158	109	149	105	156	107	151	104	154	105
128	143	113	130	126	141	115	132	124	139	117	134	122	137	119	136

. . (HG)

Besides the properties expressed in the title of this new perfect magic square; every four numbers round the centre of any component square, or at the two opposite sides, or at the four angles, always give the same amount 514, which is also the result of each corresponding row, whether horizontal, vertical, or diagonal. Every four adjacent numbers round any point, wherever taken, amount to 514. The properties of the whole square remain when its component squares are removed in vertical or horizontal rows from one side to the other; and it will continue perfectly magical, when any number of its single vertical, or horizontal rows are subjected to a like displacement, as already explained in case of the smaller square (AB), to which it bears constant analogy, but with increased variety in respect to the number and possible location of its magic components.

It will be observed that Mr. Dalby has left no theoretical construction of his magic square; and that the mode by which Dr. Hutton obtains it in his Dictionary is indirect and limited.

His method gives but one form: mine will lead without difficulty to every form of which the arrangement is susceptible; and should it be made the basis of an increased magic cyclovolute, we might determine its varieties. The remark here made applies to the square of Dr. Franklin. A number of similar and generalized forms may be obtained by imitating the elementary squares, (*d*), (*d'*), &c.

This supplementary note has extension adequate to every purpose intended. It embraces the essential principles on which depend the construction of the magic cyclovolute, and that of the magic circle; and, in the way of generalization, exhibits no unfavourable instance of contrast. Although the author has long regarded it as a necessary appendage to his paper; and as possessing some novel interest on a curious subject; he has perhaps been rather too indifferent as to the advantages usually attributed to an early publication.

Fundamental arrangements for varieties of the Magic Cyclovolute.

FIRST CLASS.

A

1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4
1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4
1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4
1	2	8	7	3	4	6	5
8	7	1	2	6	5	3	4

B

1	8	1	8	1	8	1	8
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1
2	7	2	7	2	7	2	7
3	6	3	6	3	6	3	6
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
4	5	4	5	4	5	4	5

A'

1	2	8	7	5	6	4	3
8	7	1	2	4	3	5	6
1	2	8	7	5	6	4	3
8	7	1	2	4	3	5	6
1	2	8	7	5	6	4	3
8	7	1	2	4	3	5	6
1	2	8	7	5	6	4	3
8	7	1	2	4	3	5	6

B'

1	8	1	8	1	8	1	8
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1
2	7	2	7	2	7	2	7
5	4	5	4	5	4	5	4
3	6	3	6	3	6	3	6
4	5	4	5	4	5	4	5
6	3	6	3	6	3	6	3

(*AB*)

1	16	57	56	17	32	41	40
63	50	7	10	47	34	23	26
8	9	64	49	24	25	48	33
58	55	2	15	42	39	18	31
3	14	59	54	19	30	43	38
61	52	5	12	45	36	21	28
6	11	62	51	22	27	46	35
60	53	4	13	44	37	20	29

(*BA*)

1	58	8	63	3	60	6	61
56	15	49	10	54	13	51	12
57	2	64	7	59	4	62	5
16	55	9	50	14	53	11	52
17	42	24	47	19	44	22	45
40	31	33	26	38	29	35	28
41	18	48	23	43	20	46	21
32	39	25	34	30	37	27	36

(*A'B'*)

1	16	57	56	33	48	25	24
63	50	7	10	31	18	39	42
8	9	64	49	40	41	32	17
58	55	2	15	26	23	34	47
5	12	61	52	37	44	29	20
59	54	3	14	27	22	35	46
4	13	60	53	36	45	28	21
62	51	6	11	30	19	38	43

(*B'A'*)

1	58	8	63	5	62	4	59
56	15	49	10	52	11	53	14
57	2	64	7	61	6	60	3
16	55	9	50	12	51	13	54
33	26	40	31	37	30	36	27
24	47	17	42	20	43	21	46
25	34	32	39	29	38	28	35
48	23	41	18	44	19	45	22

(*AB'*)

1	16	57	16	17	32	41	40
63	58	7	10	47	34	28	26
8	9	64	49	24	25	48	33
58	55	2	15	42	39	18	31
5	12	61	52	21	28	45	36
59	54	3	14	43	38	19	30
4	13	60	53	20	29	44	37
62	51	6	11	46	35	22	27

(*B'A*)

1	58	8	63	3	60	6	61
56	15	49	10	54	13	51	12
57	2	64	7	59	4	62	5
16	55	9	50	14	53	11	52
33	26	40	31	35	28	38	29
24	47	17	42	22	45	19	44
25	34	32	39	27	36	30	37
48	23	41	18	46	21	43	20

(*A'B*)

1	16	57	56	33	48	25	24
63	50	7	10	31	18	39	42
8	9	64	49	40	41	32	17
58	55	2	15	26	23	34	47
3	14	59	54	35	46	27	22
61	52	5	12	29	20	37	44
6	11	62	51	38	43	30	19
60	53	4	13	28	21	36	45

(*BA'*)

1	58	8	63	5	62	4	59
56	15	49	10	52	11	53	14
57	2	64	7	61	6	60	3
16	55	9	50	12	51	13	54
17	42	24	47	21	46	20	43
40	31	33	26	36	27	37	30
41	18	48	23	45	22	44	19
32	39	25	34	28	35	29	38

SECOND CLASS.

A_2

1	3	8	6	2	4	7	5
8	6	1	3	7	5	2	4
1	3	8	6	2	4	7	5
8	6	1	3	7	5	2	4
1	3	8	6	2	4	7	5
8	6	1	3	7	5	2	4
1	3	8	6	2	4	7	5
8	6	1	3	7	5	2	4

B_2

1	8	1	8	1	8	1	8
6	3	6	3	6	3	6	3
8	1	8	1	8	1	8	1
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
5	4	5	4	5	4	5	4
7	2	7	2	7	2	7	2
4	5	4	5	4	5	4	5

A_2'

1	3	8	6	5	7	4	2
8	6	1	3	4	2	5	7
1	3	8	6	5	7	4	2
8	6	1	3	4	2	5	7
1	3	8	6	5	7	4	2
8	6	1	3	4	2	5	7
1	3	8	6	5	7	4	2
8	6	1	3	4	2	5	7

B_2'

1	8	1	8	1	8	1	8
6	3	6	3	6	3	6	3
8	1	8	1	8	1	8	1
3	6	3	6	3	6	3	6
5	4	5	4	5	4	5	4
2	7	2	7	2	7	2	7
4	5	4	5	4	5	4	5
7	2	7	2	7	2	7	2

(A_2B_2)

1	24	57	48	9	32	49	40
62	43	6	19	54	35	14	27
8	17	64	41	16	25	56	33
59	46	3	22	51	38	11	30
2	23	58	47	10	31	50	39
61	44	5	20	53	36	13	28
7	18	63	42	15	26	55	34
60	45	4	21	52	37	12	29

(B_2A_2)

1	59	8	62	2	60	7	61
48	22	41	19	47	21	42	20
57	3	64	6	58	4	63	
24	46	17	43	23	45	18	44
9	51	16	54	10	52	15	53
40	30	33	27	39	29	34	28
49	11	56	14	50	12	55	13
32	38	25	35	31	37	26	36

$(A_2'B_2')$

1	24	57	48	33	56	25	16
62	43	6	19	30	11	38	51
8	17	64	41	40	49	32	9
59	46	3	22	27	14	35	54
5	20	61	44	37	52	29	12
58	47	2	23	26	15	34	55
4	21	60	45	36	53	28	13
63	42	7	18	31	10	39	50

$(B_2'A_2')$

1	59	8	62	5	63	4	58
48	22	41	19	44	18	45	23
57	3	64	6	61	7	60	2
24	46	17	43	20	42	21	47
33	27	40	30	37	31	36	26
16	54	9	51	12	50	13	55
25	35	32	38	29	39	28	34
56	14	49	11	52	10	53	15

(A_2B_2')

1	24	57	48	9	32	49	40
32	43	6	19	54	35	14	27
8	17	64	41	16	25	56	33
59	46	3	22	51	38	11	30
5	20	61	44	13	28	53	36
58	47	2	23	50	39	10	31
4	21	60	45	12	29	52	37
63	42	7	18	55	34	15	26

$(B_2'A_2)$

1	59	8	62	2	60	7	61
48	22	41	19	47	21	42	20
57	3	64	6	58	4	63	5
24	46	17	43	23	45	18	44
33	27	40	30	34	28	39	29
16	35	17	38	15	53	10	52
25	35	32	38	26	36	31	37
56	14	49	11	55	13	50	12

$(A_2'B_2)$

1	24	57	48	33	56	25	16
62	43	6	19	30	11	38	51
8	17	64	41	40	49	32	9
59	46	3	22	27	14	35	54
2	23	58	47	34	55	26	15
61	44	5	20	29	12	37	52
7	18	63	42	39	50	31	10
60	45	4	21	28	13	36	53

(B_2A_2')

1	59	8	62	5	63	4	58
48	22	41	19	44	18	45	23
57	3	64	6	61	7	60	2
24	46	17	43	20	42	21	47
9	51	16	54	13	55	12	50
40	30	33	27	36	26	37	31
49	11	56	14	53	15	52	10
32	38	25	35	28	34	29	39

THIRD CLASS.

$\mathcal{A}_3$

1	5	8	4	2	6	7	3
8	4	1	5	7	3	2	6
1	5	8	4	2	6	7	3
8	4	1	5	7	3	2	6
1	5	8	4	2	6	7	3
8	4	1	5	7	3	2	6
1	5	8	4	2	6	7	3
8	4	1	5	7	3	2	6

B_3

1	8	1	8	1	8	1	8
4	5	4	5	4	5	4	5
8	1	8	1	8	1	8	1
5	4	5	4	5	4	5	4
2	7	2	7	2	7	2	7
3	6	3	6	3	6	3	6
7	2	7	2	7	2	7	2
6	3	6	3	6	3	6	3

$\mathcal{A}_3'$

1	5	8	4	3	7	6	2
8	4	1	5	6	2	3	7
1	5	8	4	3	7	6	2
8	4	1	5	6	2	3	7
1	5	8	4	3	7	6	2
8	4	1	5	6	2	3	7
1	5	8	4	3	7	6	2
8	4	1	5	6	2	3	7

B'_3

1	8	1	8	1	8	1	8
4	5	4	5	4	5	4	5
8	1	8	1	8	1	8	1
5	4	5	4	5	4	5	4
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
6	3	6	3	6	3	6	3
7	2	7	2	7	2	7	2

$(\mathcal{A}_3 B_3)$

1	40	57	32	9	48	49	24
60	29	4	37	52	21	12	45
8	33	64	25	16	41	56	17
61	28	5	36	53	20	13	44
2	39	58	31	10	47	50	23
59	30	3	38	51	22	11	46
7	34	63	26	15	42	55	18
62	27	6	35	54	19	14	43

$(B_3 \mathcal{A}_3)$

1	61	8	60	2	62	7	59
32	36	25	37	31	35	26	38
57	5	64	4	58	6	63	3
40	28	33	29	37	27	34	30
9	53	16	52	10	54	15	51
24	44	17	45	23	43	18	46
49	13	56	12	50	14	55	11
48	20	41	21	47	19	42	22

$(\mathcal{A}_3' B_3')$

1	40	57	32	17	56	41	16
60	29	4	37	44	13	20	53
8	33	64	25	24	49	48	9
61	28	5	36	45	12	21	52
3	38	59	30	19	54	43	14
58	31	2	39	42	15	18	55
6	35	62	27	22	51	46	11
63	26	7	34	47	10	23	50

$(B_3' \mathcal{A}_3')$

1	61	8	60	3	63	6	58
32	36	25	37	30	34	27	39
57	5	64	4	59	7	62	2
40	28	33	29	38	26	35	31
17	45	24	44	19	47	22	42
16	52	9	53	14	50	11	55
41	21	48	20	43	23	46	18
56	12	49	13	54	10	51	15

$(\mathcal{A}_3 B_3')$

1	40	57	32	9	48	49	24
60	29	4	37	52	21	12	45
8	33	64	25	16	41	56	17
61	28	5	36	53	20	13	44
3	38	59	30	11	46	51	22
58	31	2	39	50	23	10	47
6	35	62	27	14	43	54	19
63	26	7	34	55	18	15	42

$(B_3' \mathcal{A}_3)$

1	61	8	60	2	62	7	59
32	36	25	37	31	35	26	38
57	5	64	4	58	6	63	3
40	28	33	29	37	27	34	30
17	45	24	44	18	46	23	43
16	52	9	53	15	51	10	54
41	21	48	20	42	22	47	19
56	12	49	13	55	11	50	14

$(\mathcal{A}_3' B_3)$

1	40	57	32	17	56	41	16
60	29	4	37	44	13	20	53
8	33	64	25	24	49	48	9
61	28	5	36	45	12	21	52
2	39	58	31	18	55	42	15
59	30	3	38	43	14	19	54
7	34	63	26	23	50	47	10
62	27	6	35	46	11	22	51

$(B_3 \mathcal{A}_3')$

1	61	8	60	3	63	6	58
32	36	25	37	30	34	27	39
57	5	64	4	59	7	62	2
40	28	33	29	38	26	35	31
9	53	16	52	11	55	14	50
24	44	17	45	21	42	19	47
49	13	56	12	51	15	54	10
48	20	41	21	46	18	43	23